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Buchstab's Function and Irregularities in the distribution of primes

Recall $\Phi(x,z) = \#\{n \le x : P^-(n) > z\}$. By Theorem LS1,

$$(\Phi) \qquad \begin{aligned} \Phi(x, x^{1/s}) &= S^+(x,s) + S^-(x,s) \\ &= \frac{x\,\omega(s)}{\log(x^{1/s})} + O_N\left(\frac{x}{\log^2 x}\right) \end{aligned}$$

uniformly for $1 + \frac{1}{N} \le s \le N$ (N is any fixed positive integer), where

$$\omega(s) = \frac{F(s) + f(s)}{2e^{\gamma}}.$$

By the differential-delay equations for f and F, we have

$$(\omega) \qquad \omega(s) = \frac{1}{s} \quad (1 \le s \le 2), \qquad (s\,\omega(s))' = \omega(s-1) \quad (s > 2).$$

Moreover, ω has a continuous derivative for $s > 2$.

<u>Lemma ω</u>

The function $\omega(s) - e^{-\gamma}$ changes sign infinitely often.

<u>Proof</u> Recall that $\lim_{s\to\infty} \omega(s) = e^{-\gamma}$. Suppose that $\omega(s)$ has finitely many sign changes.

Let $\Delta(s) = \omega(s) - e^{-\gamma}$. By (ω),

$$(\Delta) \qquad (s\Delta(s))' = \Delta(s-1).$$

Integrating from u to ∞ gives $-u\Delta(u) = \int_{u-1}^{\infty} \Delta(s)\,ds$. Thus, if $\Delta(u) \ge 0$ for large u or $\Delta(u) \le 0$ for large u, then $\Delta(u) = 0$ for large u.

Let $u_0 = \inf\{u : \Delta(s) = 0 \text{ for } s > u\}$. By (ω), $u_0 > 2$. But (Δ) implies that $\Delta(s-1) = 0$ for $s > u_0$, contradicting the definition of u_0. ∎

<u>Remarks</u> In fact, $\omega(u) - e^{-\gamma}$ changes sign on every unit interval $[y, y+1]$ for $y \ge 1$.

Primes in short intervals

It is generally expected that $\pi(x) - \pi(x-y) \sim \frac{y}{\log x}$ if y is "not too small". Unconditionally, it is known that

$$(\pi) \qquad \pi(x) - \pi(x-y) \sim \frac{y}{\log x}$$

for $x^{\frac{7}{12}+\varepsilon} \leq y \leq x$ (Huxley, 1972). Assuming the Riemann Hypothesis, (π) holds for $x^{\frac{1}{2}+\varepsilon} \leq y \leq x$, and (Selberg, 1943) holds for "almost all" intervals $[x-y, x]$ if $y = (\log x)^{2+\varepsilon}$; that is, if $y = (\log x)^{2+\varepsilon}$, (π) holds for all $x \leq X$ except for a set of $x \leq X$ of measure $o(X)$.

By Theorem Δ, we know that (π) is false for some x and
$$y = c \log x \, \frac{\log_2 x \, \log_4 x}{(\log_3 x)^2}.$$

Cramér argued heuristically that
$$0 < \limsup_{n \to \infty} \frac{p_{n+1} - p_n}{\log^2 p_n} < \infty,$$
so that (π) shouldn't hold for $y = c \cdot (\log x)^2$.

Theorem M (H. Maier, 1985)

For every $\lambda > 1$, (π) is false for $y = (\log x)^\lambda$. More precisely,
$$\liminf_{x \to \infty} \frac{\pi(x) - \pi(x-y)}{y / \log x} < 1, \qquad \limsup_{x \to \infty} \frac{\pi(x) - \pi(x-y)}{y / \log x} > 1.$$

The proof is based on Lemma ω and Maier's "matrix method".
We also need a result on primes in progressions.

Lemma Ga (Gallagher, 1970)

For certain positive constants c_1, c_2, c_3 the following holds:
If $q \geq c_1$, and for all Dirichlet characters χ modulo q, $L(\sigma + it, \chi) \neq 0$ for $\sigma \geq 1 - \frac{c_2}{\log(q(|t|+2))}$, and $(a, q) = 1$, then
$$\pi(2x, q, a) - \pi(x, q, a) = \frac{li(2x) - li(x)}{\phi(q)} \left(1 + O\left(e^{-c_3 \frac{\log x}{\log q}} + e^{-c_3 \sqrt{\log x}} \right) \right).$$

Lemma

For infinitely many integers k, $q = \prod_{p \leq k} p$ satisfies the hypotheses of Lemma Ga, if c_2 is small enough.

<u>Proof</u> By classical results, if $c_4 > 0$ is small enough, then for all q,

$$(*) \quad L(\sigma+it,\chi) \neq 0 \quad \text{for } \sigma \geq 1 - \frac{c_4}{\log(q(|t|+2))},$$

with the possible exception of one real zero for one real χ.
Let k be large, $q = \prod_{p \le k} p$. If the exceptional zero β exists for χ mod q, then $\beta > 1 - \frac{c_4}{\log(2q)}$. Since $\log q \sim k$, there is a $k' \geq k$ so that if $q' = \prod_{p \le k'} p$, then

$$1 - \frac{c_4}{\log(2q')} < \beta < 1 - \frac{c_4/2}{\log(2q')}.$$

As χ induces a character χ' modulo q' (both with the same conductor), $L(\beta,\chi') = 0$ and hence (χ',β) is the unique pair of (character, zero) that violate (*) with q replaced by q'. Hence, with $c_2 = \frac{c_4}{2}$, the hypothesis of Lemma Ga holds either with modulus q or with q'. ∎

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<u>Proof of Theorem M</u>

Fix λ and some $\lambda_1 > \lambda$. Let k be sufficiently large such that

$$q = \prod_{p \le k} p$$

satisfies the hypothesis of Lemma Ga. Put

$$u = \lfloor k^{\lambda_1} \rfloor$$
$$D = \lfloor \log\log q \rfloor + 1$$
$$R = q^{D-1}$$
$$x = Rq = q^D.$$

Consider the matrix

$$M = \begin{pmatrix} 1+Rq & 2+Rq & \cdots & u+Rq \\ 1+(R+1)q & 2+(R+1)q & \cdots & u+(R+1)q \\ \vdots & \vdots & & \vdots \\ 1+(2R-1)R & 2+(2R-1)q & \cdots & u+(2R-1)q \end{pmatrix}$$

If k is large, then $u < q$, since $u \sim (\log q)^{\lambda_1}$ (here, by $\sim$ and $o(1)$ we mean as $k \to \infty$; u, D, R and x also $\to \infty$ as $k \to \infty$). Hence the entries of M are distinct integers in $(x, 2x]$.

Let N be the number of primes in M. By Lemma Ga,

$$N = \sum_{\substack{1\le a\le U\\(a,q)=1}} \pi(2x;q,a) - \pi(x;q,a)$$

$$= \sum_{\substack{1\le a\le U\\(a,q)=1}} \frac{li(2x)-li(x)}{\phi(q)}\left(1+O(e^{-c_3\sqrt{\log x}}+e^{-c_3\log\log q})\right)$$

$$\sim \frac{x}{\phi(q)\log x}\cdot \Phi(U,k).$$

By (Φ),
$$\Phi(U,k) \sim \frac{\omega(\lambda_1)U}{\log k},$$

and by Mertens estimate,
$$\phi(q) = q\prod_{p\le k}\left(1-\frac{1}{p}\right) \sim \frac{e^{-\gamma}q}{\log k}.$$

Therefore,
(N)
$$N \sim \frac{Ux\,e^{\gamma}\omega(\lambda_1)}{q\log x}.$$

Part I Take λ_1 so that $e^{\gamma}\omega(\lambda_1)<1$. Such λ_1 exists by Lemma ω.
There are R rows of M, each consisting of an interval of U integers.
By (N), some row has
$$\le (1+o(1))\,\frac{U\,e^{\gamma}\omega(\lambda_1)}{\log x}$$
primes. Note that
$$U \sim (\log q)^{\lambda_1} = (\log x)^{\lambda_1+o(1)}.$$
Break this row into $(U/y)+O(1)$ intervals of length $y=(\log 2x)^{\lambda}$. Then some subinterval contains at most
$$(1+o(1))\,\frac{U}{(U/y+O(1))}\,\frac{e^{\gamma}\omega(\lambda_1)}{\log x} \sim \frac{y}{\log x}\,e^{\gamma}\omega(\lambda_1)$$
primes. That is, for some $x'\in(x,2x]$,
$$\frac{\pi(x')-\pi(x'-(\log x')^{\lambda})}{(\log x')^{\lambda-1}} \le \frac{\pi(x')-\pi(x'-y)}{(\log x')^{\lambda-1}} \le (1+o(1))\,e^{\gamma}\omega(\lambda_1).$$

For large enough k the right side is <1.

Part II Take λ_1 so that $e^{\gamma}\omega(\lambda_1) > 1$. Such λ_1 exists by Lemma ω.
By (N), some row of M has

$$\geqslant (1+o(1)) \frac{u e^{\gamma}\omega(\lambda_1)}{\log x}$$

primes. Let $y = (\log x)^{\lambda}$. By an argument similar to that in Part I, some interval of length y in some row of M has

$$\geqslant (1+o(1)) \frac{y}{\log x} e^{\gamma}\omega(\lambda_1)$$

primes. Then, for some $x' \in (x, 2x]$,

$$\frac{\pi(x') - \pi(x' - (\log x')^{\lambda})}{(\log x')^{\lambda-1}} \geqslant \frac{\pi(x') - \pi(x'-y)}{(\log x')^{\lambda-1}} \geqslant (1+o(1))\, e^{\gamma}\omega(\lambda_1).$$

For large k, the right side is > 1 ∎

Primes in progressions to large moduli

Let $\Delta(x;Q) = \sum_{q \le Q} \max_{(a,q)=1} \max_{y \le x} \left|\psi(y;q,a) - \frac{y}{\phi(q)}\right|$, $\psi(y;q,a) = \sum_{\substack{n \le y \\ n \equiv a \pmod q}} \Lambda(n)$.

Theorem BV is: $\forall A>0\ \exists B$ such that

$$\Delta(x; \sqrt{x}(\log x)^{-B}) \ll \frac{x}{(\log x)^A}.$$

Elliott–Halberstam conjecture: $\forall A>0, \forall \varepsilon>0$, $\Delta(x; x^{1-\varepsilon}) \ll \frac{x}{(\log x)^A}$.

E., H. also conjecture: $\forall A>0\ \exists B>0$ so that $\Delta(x; x(\log x)^{-B}) \ll \frac{x}{(\log x)^A}$.

False!

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<u>Theorem</u> (Friedlander, Granville 1989) For all $B>0$,

$$\Delta(x; x(\log x)^{-B}) \gg_B \frac{x}{\log\log x}.$$

<u>Proof</u> Fix $\lambda > B$ with $e^{\gamma}\omega(\lambda) \neq 1$ (such λ exists by Lemma ω).
Put $z = (\log x)^{B/\lambda}$ and

$$a = \prod_{p \le z} p$$

so that $a \le \sqrt{x}$. Let $Q = x/(\log x)^B$ and consider

$$S = \sum_{\substack{\frac{Q}{2} < q \le Q \\ (a,q)=1}} \psi(x;q,a) = \sum_{\substack{\frac{Q}{2} < q \le Q \\ (a,q)=1}} \Big(\sum_{\substack{p \le x \\ p \equiv a \pmod q}} \log p \Big) + O(\sqrt{x}),$$

since for $b \geqslant 2$, $p^b - a$ has at most 1 prime factor $> Q/2$, and $p^b \le x$.

If $\frac{Q}{2}<q\le Q$ and $p\equiv a \pmod q$ is prime with $p\le x$, then

$$p=rq+a,\quad r\le \frac{2(x-a)}{Q} \text{ and } \frac{Q}{2}r+a<p\le \min(x, Qr+a).$$

Hence

$$S=\sum_{\substack{r\le \frac{2(x-a)}{Q}\\ (r,a)=1}} \psi\left(\max(x,Qr+a);r,a\right)-\psi\left(\tfrac{Q}{2}r+a;r,a\right)+O(\sqrt{x}) \quad \left(\text{use Siegel-Walfisz} \atop \log(Qr+a)=\log x+O(\log\log x)\right)$$

$$=\left(1+O\left(\frac{\log\log x}{\log x}\right)\right)\sum_{\substack{r\le \frac{2(x-a)}{Q}\\ (r,a)=1}} \frac{\max(x,Qr+a)-\left(\frac{Q}{2}r+a\right)}{\phi(r)}.$$

Now $r\le 2(\log x)^B$, hence

$$\phi(r)=r\prod_{p\mid r}\left(1-\tfrac{1}{p}\right)=re^{O\left(\frac{\omega(r)}{z}\right)}=re^{O\left(\frac{\log r}{z}\right)}=r\left(1+O\left(\tfrac{1}{\sqrt{z}}\right)\right).$$

Thus, as $x\to\infty$,

$$S\sim\sum_{\substack{r\le 2x/Q\\ (r,a)=1}} \frac{\max(x,Qr)-\frac{Q}{2}r}{r}$$

$$=\frac{Q}{2}\Phi\left(\tfrac{x}{Q},z\right)-\frac{Q}{2}\left\{\Phi\left(\tfrac{2x}{Q},z\right)-\Phi\left(\tfrac{x}{Q},z\right)\right\}+x\sum_{\substack{\frac{x}{Q}<r\le\frac{2x}{Q}\\ (r,a)=1}}\frac{1}{r}$$

$$=Q\,\Phi\left(\tfrac{x}{Q},z\right)-\frac{Q}{2}\Phi\left(\tfrac{2x}{Q},z\right)+x\left\{\frac{\Phi(t)}{t}\Big|_{t=x/Q}^{t=2x/Q}+\int_{x/Q}^{2x/Q}\frac{\Phi(t,z)}{t^2}\,dt\right\}$$

$$=\int_{x/Q}^{2x/Q}\frac{\Phi(t,z)}{t^2}\,dt.$$

Uniformly for $\frac{x}{Q}\le t\le\frac{2x}{Q}$, $\frac{\log t}{\log z}=\frac{B\log\log x+O(1)}{(B/\lambda)\log\log x}\sim\lambda$, so $\Phi(t,z)\sim\omega(\lambda)\frac{t}{\log z}$,

so that

$$S\sim x\cdot\frac{\omega(\lambda)}{\log z}\int_{x/Q}^{2x/Q}\frac{dt}{t}=\frac{x\,\omega(\lambda)\log 2}{\log z}.$$

On the other hand, we obtain an asymptotic for

$$T:=\sum_{\substack{\frac{Q}{2}<q\le Q\\ (a,q)=1}}\frac{x}{\phi(q)}.$$

Break the sum into two parts: T_1, the sum over q with $\omega(q)\ge 10\log\log Q$, and T_2, the sum over those q with $\omega(q)\le 10\log\log Q$.

By Proposition C,

$$\#\{Q/2 < q \le Q : \omega(q) \ge 10 \log\log Q\} \ll \frac{Q}{(\log Q)^{5-\sqrt{e}}} \ll \frac{Q}{(\log x)^3}.$$

Hence

$$T_1 \ll \sum_{\substack{Q/2 < q \le Q \\ \omega(q) \ge 10 \log\log Q}} \frac{x \log\log Q}{Q} \ll \frac{x}{(\log x)^2}.$$

Also, for $(q,a)=1$, $P^-(q) > Z$, so for $\omega(q) \le 10 \log\log q$,

$$\phi(q) = q \prod_{p|q}\left(1 - \frac{1}{p}\right) = q\, e^{O\left(\frac{\omega(q)}{Z}\right)} = q\left(1 + O\left(\frac{\log\log x}{Z}\right)\right) \sim q,$$

so that

$$T_2 \sim x \sum_{\substack{Q/2 < q \le Q \\ (a,q)=1 \\ \omega(q) < 10\log\log q}} \frac{1}{q} = x \sum_{\substack{\frac{Q}{2} < q \le Q \\ (a,q)=1}} \frac{1}{q} + O\left(\frac{x}{\log^3 x}\right)$$

$$\sim x\left[\frac{\Phi(t,Z)}{t}\Bigg|_{Q/2}^{Q} + \int_{Q/2}^{Q} \frac{\Phi(t,Z)}{t^2}\,dt\right] \qquad \frac{\log t}{\log Z} \gg \frac{\log x}{\log\log x} \to \infty$$

$$\sim x \int_{Q/2}^{Q} \frac{e^{-\gamma}/\log Z}{t}\,dt \sim \frac{x e^{-\gamma} \log 2}{\log Z}.$$

Therefore,

$$\Delta\left(x; \frac{x}{(\log x)^B}\right) \ge \sum_{\substack{\frac{Q}{2} < q \le Q \\ (a,q)=1}} \left|\psi(x;q,a) - \frac{x}{\phi(q)}\right|$$

$$\ge \left|\sum_{\substack{\frac{Q}{2} < q \le Q \\ (a,q)=1}} \left(\psi(x;q,a) - \frac{x}{\phi(q)}\right)\right|$$

$$\sim \frac{x \log 2}{\log Z}\left|\omega(\lambda) - e^{-\gamma}\right| \gg_B \frac{x}{\log\log x} \blacksquare$$

<u>Theorem</u> (Friedlander, Granville, Hildebrand, Maier 1991)
$\forall A > 0$, if $Q = x \exp\left\{-\frac{A}{2} \frac{(\log_2 x)^2}{\log_3 x}\right\}$, then

$$\Delta(x;Q) \gg \frac{x}{(\log x)^A}.$$